Lecture 20
The Variable Eddington Factor Method

1 Introduction

The purpose of this lecture is to describe a method for solving the transport equation that is perhaps the most popular method in the astrophysical community. The method is called the variable Eddington factor (VEF) method. This method can be cast as a non-linear variant of diffusion-synthetic acceleration, but it is most versatile when used in different context.

2 Derivation of the VEF Method

We begin the derivation of the VEF method by considering the monoenergetic transport equation with isotropic scattering and distributed sources:

$$\mu\frac{\partial\psi}{\partial x} + \sigma_t\psi = \frac{\sigma_s}{4\pi}\phi + \frac{q_0}{4\pi}\,. \tag{1}$$

As discussed in a previous lecture, taking the zero'th and first angular moments of the transport equation yields the particle balance equation and the momentum balance equation, respectively:

$$\frac{\partial J}{\partial x} + \sigma_s \phi = q_0 \,, \tag{2}$$

$$\frac{\partial P}{\partial x} + \sigma_t J = 0 \,, \tag{3}$$

where P is the radiation pressure:

$$P = 2\pi \int_{-1}^{+1} \mu^2 \psi \, d\mu \,. \tag{4}$$

This system of equations is not closed because there are two equations and three unknowns. In the VEF method, this system is closed via a quantity called the variable Eddington factor, f, which is defined as the ratio of the pressure to the scalar flux:

$$f = P/\phi \,. \tag{5}$$

Substituting from Eq. (5) into Eq. (3), and using the resulting equation to eliminate the current from Eq. (2), we get an equation for the scalar flux:

$$-\frac{\partial}{\partial x}\frac{1}{\sigma_t}\frac{\partial}{\partial x}\left(f\phi\right) + \sigma_a \phi = q_0 \,. \tag{6}$$

Expanding out the derivative term , we find that Eq. (6) is a drift-diffusion equation:

$$-\frac{\partial}{\partial x}\frac{f}{\sigma_t}\frac{\partial \phi}{\partial x} - \frac{\partial}{\partial x}\frac{\phi}{\sigma_t}\frac{\partial f}{\partial x} + \sigma_a \phi = q_0 \,. \tag{7}$$

The central theme behind the VEF method is first use a transport sweep with a lagged scattering source to provide an estimate of the VEF factor, and then solve the moment equations to obtain a new scattering source estimate. This process is repeated until convergence is achieved. This two-step VEF iteration scheme can be represented as follows. First a sweep,

$$\mu\frac{\partial\psi^{\ell+\frac{1}{2}}}{\partial x}+\sigma_t\psi^{\ell+\frac{1}{2}}=\frac{\sigma_s}{4\pi}\phi^{\ell}+\frac{q_0}{4\pi}\,, \tag{8}$$

then a calculation of the variable Eddington factor,

$$f^{\ell+\frac{1}{2}}=\frac{P^{\ell+\frac{1}{2}}}{\phi^{\ell+\frac{1}{2}}}=\frac{2\pi\int_{-1}^{+1}\mu^2\psi^{\ell+\frac{1}{2}}\,d\mu}{2\pi\int_{-1}^{+1}\psi^{\ell+\frac{1}{2}}\,d\mu}\,. \tag{9}$$

then the solution of the second-order scalar flux equation,

$$-\frac{\partial}{\partial x}\frac{1}{\sigma_t}\frac{\partial}{\partial x}\left(f^{\ell+\frac{1}{2}}\phi^{\ell+1}\right)+\sigma_a\phi^{\ell+1}=q_0\,. \tag{10}$$

This iteration scheme converges with roughly the same rate as convergence as the DSA method. It has several advantages relative to the DSA method.

1. The second-order VEF scalar flux equation need not be differenced in a manner that is consistent with the spatial differencing of the transport equation. The convergence rate does not depend upon consistency. However, upon convergence, the scalar flux obtained from the second-order equation will not be identical to that obtained from the transport equation unless there is consistent differencing. In the inconsistent case, the best scalar flux to use is that from the second-order equation because the second-order solution is conservative, whereas the transport solution is not conservative.

2. Spatial discretizations for the tranport equation that would not normally have the thick diffusion limit can nonetheless have the thick diffusion limit when used in the VEF algorithm. To obtain the diffusion limit, one need only ensure that the variable Eddington factor goes to $\frac{1}{3}$ in the diffusion limit. This is a much simpler requirement than one generally has with the S_n method.

3. There are certain non-conservative transport discretization schemes (based upon short characteristic methods rather than the S_n method) that yield good angular flux shapes, but poor angular flux magnitudes. Such schemes work well in conjunction with the VEF method.

The VEF method also has several disadvantages relative to the S_n method.

1. The VEF equation for the scalar flux is a drift-diffusion equation. Such equations are generally quite difficult to discretized and iteratively solve.

2. Negative angular fluxes can cause the effective diffusion coefficient in the second-order scalar flux equation to become negative, resulting in a an unstable equation. It is very difficult on complex multidimensional meshes to difference the VEF scalar flux equation so that it will yield strictly positive solutions given positive sources. The DSA method is not affected by negativities. Thus it would appear that DSA is more robust than the VEF method on complex meshes. However, this question merits

further investigation.

3. The VEF scalar flux equation is singular in a void. The DSA method can be made to work well with imbedded voids simply by using a relatively small diffusion coefficient.

In the multidimensional case, the Variable Eddington factor becomes a Variable Eddington tensor, i.e.,

$$\overrightarrow{\overrightarrow{f}} = \frac{\overrightarrow{\overrightarrow{P}}}{\phi} . \tag{11}$$